8. Problem sheet for Set Theory, Winter 2012

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Problem 27. Prove:

(a) If α is an ordinal, then $TC(\alpha) = \alpha$ and $TC(\{\alpha\}) = \alpha + 1$.

(b) If $(X, <)$ is a well-ordered set and $Y \subseteq X$, then $otp(Y, <) \leq otp(X, <)$, where *otp* denotes the order type.

(c) If α is a countable limit ordinal, then there is a strictly increasing sequence $(\beta_i \mid i \in \omega)$ with $\alpha = \sup_{i \in \omega} \beta_i$.

Problem 28. Define $+\colon V \times V \to V$ and $\cdot\colon V \times V \to V$ by

- $x + y = x \cup \{x + z \mid z \in y\}$ and
- $x \cdot y = \bigcup_{z \in y}(x \cdot z + x)$.

Show:

(a) $+$ and $\cdot$ extend ordinal addition and multiplication.

(b) $x + (y + z) = (x + y) + z$ for all $x, y, z \in V$.

Problem 29. Show:

(a) The set of algebraic numbers is countable.

(b) The set of transcendental numbers has the same cardinality as $\mathbb{R}$.

(c) The set of open subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$.

Problem 30. A set X is called *Dedekind infinite* if there is an injective but not surjective function $f\colon X \to X$.

(a) Prove in ZF that every Dedekind infinite set is infinite.

(b) Work in ZF plus the *Axiom of countable choice*: $\forall x(\overline{\overline{x}} \leq \omega \to \exists g$ (g is a function with domain $x \wedge \forall u \in x(u \neq \emptyset \to g(u) \in u)))$. Suppose X is infinite. Let $f(n)$ denote the set of all injective functions $t\colon n \to X$ for $n \in \omega$. Show that $f(n)$ is nonempty for each n. Show that X is Dedekind infinite.

There are 6 points for each problem. Please hand in your solutions on Monday, December 3 before the lecture.